

# $L^2$ -SERRE DUALITY ON SINGULAR COMPLEX SPACES AND APPLICATIONS

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**ABSTRACT.** In this survey, we explain a version of topological  $L^2$ -Serre duality for singular complex spaces with arbitrary singularities. This duality can be used to deduce various  $L^2$ -vanishing theorems for the  $\bar{\partial}$ -equation on singular spaces. As one application, we prove Hartogs' extension theorem for  $(n-1)$ -complete spaces. Another application is the characterization of rational singularities. It is shown that complex spaces with rational singularities behave quite tame with respect to some  $\bar{\partial}$ -equation in the  $L^2$ -sense. More precisely: a singular point is rational if and only if the appropriate  $L^2$ - $\bar{\partial}$ -complex is exact in this point. So, we obtain an  $L^2$ - $\bar{\partial}$ -resolution of the structure sheaf in rational singular points.

## 1. INTRODUCTION

Classical Serre duality, [S1], can be formulated as follows: Let  $X$  be a complex  $n$ -dimensional manifold, let  $V \rightarrow X$  be a complex vector bundle, and let  $\mathcal{E}^{0,q}(X, V)$  and  $\mathcal{E}_c^{n,q}(X, V^*)$  be the spaces of global smooth  $(0, q)$ -form with values in  $V$  and global smooth compactly supported  $(n, q)$ -forms with values in the dual bundle  $V^*$ , respectively. Then the following pairing is non-degenerate

$$H^q(\mathcal{E}^{0,\bullet}(X, V), \bar{\partial}) \times H^{n-q}(\mathcal{E}_c^{n,\bullet}(X, V^*), \bar{\partial}) \rightarrow \mathbb{C}, \quad ([\varphi]_{\bar{\partial}}, [\psi]_{\bar{\partial}}) \mapsto \int_X \varphi \wedge \psi \quad (1)$$

provided that  $H^q(\mathcal{E}^{0,\bullet}(X, V), \bar{\partial})$  and  $H^{q+1}(\mathcal{E}^{0,\bullet}(X, V), \bar{\partial})$  are Hausdorff topological vector spaces.

If  $X$  is allowed to have singularities, then, traditionally, Serre duality takes a more algebraic and much less explicit form. To explain that more precisely, let  $\mathcal{F} := \mathcal{O}(F)$ ,  $\mathcal{F}^* := \mathcal{O}(F^*)$  and let  $\Omega_X^n$  denote the sheaf of holomorphic  $n$ -forms on  $X$ . Then we can rephrase (1) via the Dolbeault isomorphism algebraically: There is a non-degenerate topological pairing

$$H^q(X, \mathcal{F}) \times H^{n-q}(X, \mathcal{F}^* \otimes \Omega_X^n) \rightarrow \mathbb{C}, \quad (2)$$

realized by the cup-product, provided that  $H^q(X, \mathcal{F})$  and  $H^{q+1}(X, \mathcal{F})$  are Hausdorff. In this formulation, Serre duality has been generalized to singular complex spaces, see, e.g., Hartshorne [H1], [H2] and Conrad [C] for the algebraic setting and Ramis-Ruget [RR] and Andreotti-Kas [AK] for the analytic setting. In fact, if  $X$  is of pure dimension  $n$ , paracompact and Cohen-Macaulay, then there is again a non-degenerate topological pairing (2) if we replace  $\Omega_X^n$  by the Grothendieck dualizing sheaf  $\omega_X$ . If  $X$  is not Cohen-Macaulay, then  $H^{n-q}(X, \mathcal{F}^* \otimes \Omega_X^n)$  has to be replaced by the cohomology of a certain complex of  $\mathcal{O}_X$ -modules, called a dualizing complex.

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In this survey, we will explain how  $L^2$ -theory for the  $\bar{\partial}$ -operator can be used to obtain an  $L^2$ -version of Serre duality on singular spaces which has an analytic realization completely analogous to (1). More precisely, we will show how (1) generalizes to singular spaces by replacing the Dolbeault cohomology groups of smooth  $(0, q)$  and  $(n, q)$ -forms, respectively, by  $L^2$ -Dolbeault cohomology groups.

## 2. $L^2$ -THEORY FOR THE $\bar{\partial}$ -OPERATOR ON SINGULAR SPACES

The Cauchy-Riemann operator  $\bar{\partial}$  plays a fundamental role in Complex Analysis and Complex Geometry. On complex manifolds, functions – or more generally distributions – are holomorphic if and only if they are in the kernel of the  $\bar{\partial}$ -operator, and the same holds in a certain sense on normal complex spaces. For forms of arbitrary degree, the importance of the  $\bar{\partial}$ -operator appears strikingly for example in the notion of  $\bar{\partial}$ -cohomology which can be used to represent the cohomology of complex manifolds by the Dolbeault isomorphism.

The  $L^2$ -theory for the  $\bar{\partial}$ -operator is of particular importance in Complex Analysis and Geometry and has become indispensable for the subject after the fundamental work of Hörmander on  $L^2$ -estimates and existence theorems for the  $\bar{\partial}$ -operator [H3] and the related work of Andreotti and Vesentini [AV]. Important applications of the  $L^2$ -theory are e.g. the Ohsawa-Takegoshi extension theorem [OT], Siu's analyticity of the level sets of Lelong numbers [S2] or the invariance of plurigenera [S3] – just to name some.

The first problem one has to face when studying the  $\bar{\partial}$ -equation on singular spaces is that it is not clear what kind of differential forms and operators one should consider. Recently, there has been considerable progress by different approaches.

Andersson and Samuelsson developed in [AS] Koppelman integral formulas for the  $\bar{\partial}$ -equation on arbitrary singular complex spaces which allow for a  $\bar{\partial}$ -resolution of the structure sheaf in terms of certain fine sheaves of currents, called  $\mathcal{A}$ -sheaves. These  $\mathcal{A}$ -sheaves are defined by an iterative procedure of repeated application of singular integral operators, which makes them pretty abstract and hard to understand (and difficult to work with in concrete situations).

A second, more explicit approach is as follows: Consider differential forms which are defined on the regular part of a singular variety and which are square-integrable up to the singular set. This setting seems to be very fruitful and has some history by now (see [PS]).<sup>1</sup> Also in this direction, considerable progress has been made recently. Øvrelid–Vassiliadou and the author obtained in [OV2] and [R3] a pretty complete description of the  $L^2$ -cohomology of the  $\bar{\partial}$ -operator (in the sense of distributions) at isolated singularities.

In this setting, we understand the class of objects with which we deal very well (just  $L^2$ -forms), but the disadvantage is a different one. Whereas the  $\bar{\partial}$ -equation is locally solvable for closed  $(0, q)$ -forms in the category of  $\mathcal{A}$ -sheaves by the Koppelman formulas in [AS], there are local obstructions to solving the  $\bar{\partial}$ -equation in the  $L^2$ -sense at singular points (see e.g. [FOV], [OV2], [R3]). So, there can be no  $L^2$ - $\bar{\partial}$ -resolution for the structure sheaf in general.

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<sup>1</sup> The interest in this setting goes back to the invention of intersection (co-)homology by Goresky and MacPherson which has very tight connections to the  $L^2$ -deRham cohomology of the regular part of a singular variety. We refer here to the solution of the Cheeger-Goresky-MacPherson conjecture [CGM] for varieties with isolated singularities by Ohsawa [O] (see [PS] for more details).

In this survey, we will see that the  $\bar{\partial}$ -operator in the  $L^2$ -sense behaves pretty well on spaces with canonical singularities which play a prominent role in the minimal model program. The underlying idea is that canonical Gorenstein singularities are rational (see e.g. [K], Theorem 11.1), i.e., we expect that the singularities do not contribute to the local cohomology.

Pursuing this idea, it turned out that there is a notion of  $L^2$ - $\bar{\partial}$ -cohomology for  $(0, q)$ -forms which can be described completely in terms of a resolution of singularities (see (6) below). A singular point is rational if and only if this certain  $L^2$ - $\bar{\partial}$ -complex is exact in this point. If the underlying space has rational singularities, particularly on a Gorenstein space with canonical singularities, then we obtain an  $L^2$ - $\bar{\partial}$ -resolution of the structure sheaf, i.e., a resolution of the structure sheaf in terms of a well-known and easy to handle class of differential forms. One of our main tools is a version of topological  $L^2$ -Serre duality for singular complex spaces with arbitrary singularities, which seems to be useful in other contexts, too (Theorem 4.1).

### 3. TWO $\bar{\partial}$ -COMPLEXES ON SINGULAR COMPLEX SPACES

We need to specify what we mean by differential forms and the  $\bar{\partial}$ -operator in the presence of singularities. Let  $X$  be a Hermitian complex space<sup>2</sup> of pure dimension  $n$  and  $F \rightarrow X$  a Hermitian holomorphic line bundle. We denote by  $\mathcal{L}^{p,q}(F)$  the sheaf of germs of  $F$ -valued  $(p, q)$ -forms on the regular part of  $X$  which are square-integrable on  $K^* = K \setminus \text{Sing } X$  for any compact set  $K$  in their domain of definition.<sup>3</sup> Note that  $\mathcal{L}^{p,q}(F)$  becomes a Fréchet sheaf with the  $L^{2,loc}$ -topology on open subsets of  $X$ .

Due to the incompleteness of the metric on  $X^* = X \setminus \text{Sing } X$ , there are different reasonable definitions of the  $\bar{\partial}$ -operator on  $\mathcal{L}^{p,q}(F)$ -forms. To be more precise, let  $\bar{\partial}_{cpt}$  be the  $\bar{\partial}$ -operator on smooth forms with support away from the singular set  $\text{Sing } X$ . Then  $\bar{\partial}_{cpt}$  can be considered as a densely defined operator  $\mathcal{L}^{p,q}(F) \rightarrow \mathcal{L}^{p,q+1}(F)$ . One can now consider various closed extensions of this operator. The two most important are the maximal closed extension, i.e., the  $\bar{\partial}$ -operator in the sense of distributions which we denote by  $\bar{\partial}_w$ , and the minimal closed extension, i.e., the closure of the graph of  $\bar{\partial}_{cpt}$  which we denote by  $\bar{\partial}_s$ . Let  $\mathcal{C}^{p,q}(F)$  be the domain of definition of  $\bar{\partial}_w$  which is a subsheaf of  $\mathcal{L}^{p,q}(F)$ , and  $\mathcal{F}^{p,q}(F)$  the domain of definition of  $\bar{\partial}_s$  which in turn is a subsheaf of  $\mathcal{C}^{p,q}(F)$ . We obtain complexes of fine sheaves

$$\mathcal{C}^{p,0}(F) \xrightarrow{\bar{\partial}_w} \mathcal{C}^{p,1}(F) \xrightarrow{\bar{\partial}_w} \mathcal{C}^{p,2}(F) \xrightarrow{\bar{\partial}_w} \dots \quad (3)$$

and

$$\mathcal{F}^{p,0}(F) \xrightarrow{\bar{\partial}_s} \mathcal{F}^{p,1}(F) \xrightarrow{\bar{\partial}_s} \mathcal{F}^{p,2}(F) \xrightarrow{\bar{\partial}_s} \dots \quad (4)$$

We refer to [R4] for more details, but let us explain the  $\bar{\partial}_s$ -operator more precisely for convenience of the reader. Let  $f$  be a germ in  $\mathcal{C}^{p,q}(F)$ , i.e., an  $F$ -valued  $(p, q)$ -form on an open set  $U$  in  $X$  (living on the regular part of  $U$ ) which is  $L^2$  on compact subsets of  $U$  and such that the  $\bar{\partial}$  in the sense of distributions,  $\bar{\partial}_w f$ , is in the same class of forms. Then  $f$  is in the domain of the  $\bar{\partial}_s$ -operator (and we set  $\bar{\partial}_s f = \bar{\partial}_w f$ )

<sup>2</sup>A Hermitian complex space  $(X, g)$  is a reduced complex space  $X$  with a metric  $g$  on the regular part such that the following holds: If  $x \in X$  is an arbitrary point there exists a neighborhood  $U = U(x)$  and a biholomorphic embedding of  $U$  into a domain  $G$  in  $\mathbb{C}^N$  and an ordinary smooth Hermitian metric in  $G$  whose restriction to  $U$  is  $g|_U$ .

<sup>3</sup>This is what we mean by square-integrable up to the singular set.

if there exists a sequence of forms  $\{f_j\}_j \subset \mathcal{C}^{p,q}(U, F)$  with support away from the singular set,  $\text{supp } f_j \cap \text{Sing } X = \emptyset$ , such that

$$\begin{aligned} f_j &\rightarrow f & \text{in } \mathcal{L}^{p,q}(U, F), \\ \bar{\partial}_w f_j &\rightarrow \bar{\partial}_w f & \text{in } \mathcal{L}^{p,q+1}(U, F). \end{aligned}$$

This means that the  $\bar{\partial}_s$ -operator comes with a certain Dirichlet boundary at the singular set of  $X$ , which can also be interpreted as a growth condition. We have e.g. the following:

**Lemma 3.1** ([R4]). *Bounded forms in the domain of  $\bar{\partial}_w$  are in the domain of  $\bar{\partial}_s$ .*

If  $F$  is just the trivial line bundle, then  $\mathcal{K}_X := \ker \bar{\partial}_w \subset \mathcal{C}^{n,0}$  is the canonical sheaf of Grauert–Riemenschneider (see [GR]) and  $\mathcal{K}_X^s := \ker \bar{\partial}_s \subset \mathcal{F}^{n,0}$  is the sheaf of holomorphic  $n$ -forms with Dirichlet boundary condition that was introduced in [R3]. We will see below that  $\widehat{\mathcal{O}}_X = \ker \bar{\partial}_s \subset \mathcal{F}^{0,0}$  for the sheaf of weakly holomorphic functions  $\widehat{\mathcal{O}}_X$ .

It is clear that (3) and (4) are exact in regular points of  $X$ . Exactness in singular points is equivalent to the difficult problem of solving  $\bar{\partial}$ -equations locally in the  $L^2$ -sense at singularities, which is not possible in general (see e.g. [FOV], [OV1], [OV2], [R1], [R2], [R3]). However, it is known that (3) is exact for  $p = n$  (see [PS]), and that (4) is exact for  $p = n$  if  $X$  has only isolated singularities (see [R3]). In these cases, the complexes (3) and (4) are fine resolutions of the canonical sheaves  $\mathcal{K}_X$  and  $\mathcal{K}_X^s$ , respectively.

For an open set  $\Omega \subset X$ , we denote by  $H_{w,loc}^{p,q}(\Omega, F)$  the cohomology of the complex (3), and by  $H_{w,cpt}^{p,q}(\Omega, F)$  the cohomology of (3) with compact support. Analogously, let  $H_{s,loc}^{p,q}(\Omega, F)$  and  $H_{s,cpt}^{p,q}(\Omega, F)$  be the cohomology groups of (4). These  $L^2$ -cohomology groups inherit the structure of topological vector spaces, which are locally convex Hausdorff spaces if the corresponding  $\bar{\partial}$ -operators have closed range.<sup>4</sup>

#### 4. $L^2$ -SERRE DUALITY

We can now formulate the  $L^2$ -version of (1) for singular complex spaces:

**Theorem 4.1 (Serre duality [R4]).** *Let  $X$  be a Hermitian complex space of pure dimension  $n$ ,  $F \rightarrow X$  a Hermitian holomorphic line bundle, and let  $0 \leq p, q \leq n$ . If  $H_{w,loc}^{p,q}(\Omega, F)$  and  $H_{w,loc}^{p,q+1}(\Omega, F)$  are Hausdorff, then the mapping*

$$\mathcal{L}^{p,q}(\Omega, F) \times \mathcal{L}_{cpt}^{n-p,n-q}(\Omega, F^*) \rightarrow \mathbb{C} \quad , \quad (\eta, \omega) \mapsto \int_{\Omega^*} \eta \wedge \omega,$$

*induces a non-degenerate pairing of topological vector spaces*

$$H_{w,loc}^{p,q}(\Omega, F) \times H_{s,cpt}^{n-p,n-q}(\Omega, F^*) \rightarrow \mathbb{C}$$

*such that  $H_{s,cpt}^{n-p,n-q}(\Omega, F^*)$  is the topological dual of  $H_{w,loc}^{p,q}(\Omega, F)$  and vice versa.*

*The same statement holds with the indices  $\{s, w\}$  in place of  $\{w, s\}$ . Then there is a non-degenerate pairing*

$$H_{s,loc}^{p,q}(\Omega, F) \times H_{w,cpt}^{n-p,n-q}(\Omega, F^*) \rightarrow \mathbb{C}.$$

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<sup>4</sup> Note that different Hermitian metrics lead to  $\bar{\partial}$ -complexes which are equivalent on relatively compact subsets. So, one can put any Hermitian metric on  $X$  in many of the results below.

If the topological vector spaces  $H_{w/s,loc}^{p,q}(\Omega, F)$ ,  $H_{w/s,loc}^{p,q+1}(\Omega, F)$  are non-Hausdorff, then the statement of Theorem 4.1 holds at least for the separated cohomology groups  $\overline{H}_{w/s} = \ker \overline{\partial}_{w/s} / \overline{\text{Im } \overline{\partial}_{w/s}}$ .<sup>5</sup> The two main difficulties in the proof of Theorem 4.1 are as follows. First, the  $\overline{\partial}$ -operators under consideration are just closed densely defined operators in the Fréchet spaces  $\mathcal{L}^{p,q}(\Omega, F)$  and the  $(LF)$ -spaces  $\mathcal{L}_{cpt}^{n-p,n-q}(\Omega, F^*)$ . Second, we have to show that the operators  $\overline{\partial}_w$  and  $\overline{\partial}_s$  are topologically dual, even at singularities. Note that  $H_{w/s,loc}^{p,q}(\Omega, F)$  is Hausdorff if and only if  $\overline{\partial}_{w/s}$  has closed range in  $\mathcal{L}^{p,q}(\Omega, F)$ , and to decide whether this is the case is usually as difficult as solving the corresponding  $\overline{\partial}$ -equation. Using local  $L^2$ - $\overline{\partial}$ -solution results for singular spaces, one can show at least:

**Theorem 4.2** ([R4]). *Let  $X$  be a Hermitian complex space of pure dimension  $n$ ,  $F \rightarrow X$  a Hermitian holomorphic line bundle, and let  $0 \leq p, q \leq n$ . Let  $\Omega \subset X$  be a holomorphically convex open subset. Then the topological vector spaces*

$$H_{w,loc}^{n,q}(\Omega, F) \quad , \quad H_{w,cpt}^{n,q}(\Omega, F) \quad , \quad H_{s,cpt}^{0,n-q}(\Omega, F^*) \quad , \quad H_{s,loc}^{0,n-q}(\Omega, F^*)$$

*are Hausdorff for all  $0 \leq q \leq n$ .*

A main point in the proof of Theorem 4.2 is to show that the canonical Fréchet sheaf structure of compact convergence on the coherent analytic canonical sheaf  $\mathcal{K}_X$  coincides with the Fréchet sheaf structure of  $L^2$ -convergence on compact subsets. This allows then to show also the topological equivalence of Čech cohomology and  $L^2$ -cohomology. If  $X$  has only isolated singularities, then the Hausdorff property is known also for some cohomology spaces of different degree (see [R4]).

As a direct application of Serre duality, Theorem 4.1, one can deduce:

**Theorem 4.3.** *Let  $X$  be a Hermitian complex space of pure dimension  $n$ ,  $F \rightarrow X$  a Hermitian holomorphic line bundle and  $\Omega \subset X$  a cohomologically  $q$ -complete open subset,  $q \geq 1$ . Then*

$$H_{w,loc}^{n,r}(\Omega, F) = H_{s,cpt}^{0,n-r}(\Omega, F^*) = 0 \quad \text{for all } r \geq q.$$

Note that  $\Omega$  is cohomologically  $q$ -complete if it is  $q$ -complete by the Andreotti-Grauert vanishing theorem [AG]. So, Theorem 4.3 allows to solve the  $\overline{\partial}_s$ -equation with compact support for  $(0, n-q)$ -forms on  $q$ -complete spaces, which is of particular interest for 1-complete spaces, i.e., Stein spaces.

## 5. HARTOGS' EXTENSION THEOREM

Let us mention some applications. As an interesting consequence of Theorem 4.3, we obtain Hartogs' extension theorem in its most general form. This version of the Hartogs' extension was first obtained by Merker-Porten [MP] and shortly thereafter also by Coltoiu-Ruppenthal [CR]. Merker and Porten gave an involved geometrical proof by using a finite number of parameterized families of holomorphic discs and Morse-theoretical tools for the global topological control of monodromy, but no  $\overline{\partial}$ -theory. Shortly after that, Coltoiu and Ruppenthal were able to give a short  $\overline{\partial}$ -theoretical proof by the Ehrenpreis- $\overline{\partial}$ -technique (cf. [CR]). This approach involves Hironaka's resolution of singularities which may be considered a very deep theorem. In the present survey, we give a very short proof of the extension theorem by the Ehrenpreis- $\overline{\partial}$ -technique without needing a resolution of singularities. We just use the vanishing result  $H_{s,cpt}^{0,1}(X) = 0$

<sup>5</sup>The notation  $w/s$  refers either to the index  $w$  or the index  $s$  in the whole statement.

**Theorem 5.1.** *Let  $X$  be a connected normal complex space of dimension  $n \geq 2$  which is cohomologically  $(n - 1)$ -complete. Furthermore, let  $D$  be a domain in  $X$  and  $K \subset D$  a compact subset such that  $D \setminus K$  is connected. Then each holomorphic function  $f \in \mathcal{O}(D \setminus K)$  has a unique holomorphic extension to the whole set  $D$ .*

*Proof.* Let  $f \in \mathcal{O}(D \setminus K)$ . Choose a cut-off function  $\chi \in C_{cpt}^\infty(D)$  such that  $\chi$  is identically 1 in a neighborhood of  $K$ . Then  $g := (1 - \chi)f$  is an extension of  $f$ , but unfortunately not holomorphic. However, we can fix it by the  $\bar{\partial}$ -strategy of Ehrenpreis. By Lemma 3.1,  $g$  is in the domain of  $\bar{\partial}_s$  and  $H_{s,cpt}^{0,1}(X) = 0$  by Theorem 4.3. So, there exists a solution  $h$  to the  $\bar{\partial}_s$ -equation with compact support  $\bar{\partial}_s h = \bar{\partial}_s g$  and  $F := g - h$  is the desired extension of  $f$  to the whole of  $D$ . That can be seen by use of the identity theorem and the fact that  $X$  cannot be compact (because Theorem 4.3 implies also that  $H_{s,cpt}^{0,0}(X) = 0$ ).  $\square$

## 6. RATIONAL SINGULARITIES

Another, very interesting application of  $L^2$ -Serre duality is the following characterization of rational singularities. Let  $\pi : M \rightarrow X$  be a resolution of singularities and  $\Omega \subset\subset X$  holomorphically convex. Give  $M$  any Hermitian metric. Then pullback of  $L^2$ -( $n, q$ )-forms under  $\pi$  induces an isomorphism

$$\pi^* : H_{w,cpt}^{n,q}(\Omega) \xrightarrow{\cong} H_{w,cpt}^{n,q}(\pi^{-1}(\Omega)) \cong H_{cpt}^q(\pi^{-1}(\Omega), \mathcal{K}_M) \quad (5)$$

for all  $0 \leq q \leq n$  by use of Pardon–Stern [PS] and the Takegoshi vanishing theorem [T] (see [R4] for more details). Now we can use the  $L^2$ -Serre duality, Theorem 4.1, and classical Serre duality on the smooth manifold  $\pi^{-1}(\Omega)$  to deduce that push-forward of forms under  $\pi$  induces another isomorphism

$$\pi_* : H^{n-q}(\pi^{-1}(\Omega), \mathcal{O}_M) \xrightarrow{\cong} H_{s,loc}^{0,n-q}(\Omega) \quad (6)$$

for all  $0 \leq q \leq n$  (see [R4], Theorem 1.1). This shows that the obstructions to solving the  $\bar{\partial}_s$ -equation locally for  $(0, q)$ -forms can be expressed in terms of a resolution of singularities. For the cohomology sheaves of the complex  $(\mathcal{F}^{0,\bullet}, \bar{\partial}_s)$ , we see that

$$(\mathcal{H}^q(\mathcal{F}^{0,\bullet}, \bar{\partial}_s))_x \cong (R^q \pi_* \mathcal{O}_M)_x$$

in any point  $x \in X$  for all  $q \geq 0$ . It follows that the functions in the kernel of  $\bar{\partial}_s$  are precisely the weakly holomorphic functions, and for  $p = 0$  the complex (4) is exact in a point  $x \in X$  exactly if  $x$  is a rational point:

**Theorem 6.1** ([R4], **Theorem 1.3**). *Let  $X$  be a Hermitian complex space. Then the  $L^2$ - $\bar{\partial}$ -complex*

$$0 \rightarrow \mathcal{O}_X \longrightarrow \mathcal{F}^{0,0} \xrightarrow{\bar{\partial}_s} \mathcal{F}^{0,1} \xrightarrow{\bar{\partial}_s} \mathcal{F}^{0,2} \xrightarrow{\bar{\partial}_s} \mathcal{F}^{0,3} \xrightarrow{\bar{\partial}_s} \dots \quad (7)$$

*is exact in a point  $x \in X$  if and only if  $x$  is a rational point.*

*Hence, if  $X$  has only rational singularities, then (7) is a fine resolution of the structure sheaf  $\mathcal{O}_X$ .*

Recall that a point  $x \in X$  is rational if it is a normal point and  $(R^q \pi_* \mathcal{O}_M)_x = 0$  for all  $q \geq 1$ . If  $X$  has only rational singularities, then Theorem 6.1 yields immediately further finiteness and vanishing results, e.g. if  $X$  is  $q$ -convex or  $q$ -complete.



Let us point out also the following interesting fact. Let  $X$  be a Gorenstein space with canonical singularities. By exactness of (7) and exactness of (3) for  $p = n$ , the non-degenerate  $L^2$ -Serre duality pairing

$$H_{s,loc}^{0,q}(\Omega) \times H_{w,cpt}^{n,n-q}(\Omega) \rightarrow \mathbb{C}, \quad ([\eta], [\omega]) \mapsto \int_{\Omega^*} \eta \wedge \omega,$$

is for  $0 \leq q \leq n$  then an explicit realization of Grothendieck duality after Ramis-Ruget [RR],

$$(H^q(\Omega, \mathcal{O}_X))^* \cong H_{cpt}^{n-q}(\Omega, \omega_X),$$

given the cohomology groups under consideration are Hausdorff. Here,  $\omega_X$  denotes the Grothendieck dualizing sheaf which coincides with the Grauert-Riemenschneider canonical sheaf  $\mathcal{K}_X$  as  $X$  has canonical Gorenstein singularities.

## 7. $\mathcal{A}$ -SHEAF DUALITY

We conclude by mentioning another approach to analytic Serre duality on singular complex spaces which is based on the so-called  $\mathcal{A}_{0,q}$ -sheaves introduced by Andersson and Samuelsson in [AS]. These are certain sheaves of  $(0, q)$ -currents on singular complex spaces which are smooth on the regular part of the variety and such that the  $\bar{\partial}$ -complex

$$0 \rightarrow \mathcal{O}_X \hookrightarrow \mathcal{A}_{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}_{0,1} \xrightarrow{\bar{\partial}} \mathcal{A}_{0,2} \rightarrow \dots \quad (8)$$

is a fine resolution of the structure sheaf. The  $\mathcal{A}$ -sheaves are defined via Koppelman integral formulas on singular complex spaces.

Analogously, in [RSW], we introduced a  $\bar{\partial}$ -complex of fine sheaves of  $(n, q)$ -currents (smooth on the regular part of the variety)

$$0 \rightarrow \omega_X \hookrightarrow \mathcal{A}_{n,0} \xrightarrow{\bar{\partial}} \mathcal{A}_{n,1} \xrightarrow{\bar{\partial}} \mathcal{A}_{n,2} \rightarrow \dots \quad (9)$$

where  $X$  is of pure dimension  $n$  and  $\omega_X$  denotes the Grothendieck dualizing sheaf. The complex (9) is exact only under some additional assumptions, e.g. if  $X$  is Cohen-Macaulay. We call  $(\mathcal{A}_{n,\bullet}, \bar{\partial})$  a dualizing Dolbeault complex for  $\mathcal{O}_X$  because we obtain in [RSW] a non-degenerate topological pairing

$$H^q(\mathcal{A}_{0,\bullet}(X), \bar{\partial}) \times H_{cpt}^{n-q}(\mathcal{A}_{n,\bullet}(X), \bar{\partial}) \rightarrow \mathbb{C}, \quad ([\varphi]_{\bar{\partial}}, [\psi]_{\bar{\partial}}) \mapsto \int_X \varphi \wedge \psi, \quad (10)$$

provided that  $H^q(X, \mathcal{O}_X) \cong H^q(\mathcal{A}_{0,\bullet}(X), \bar{\partial})$  and  $H^{q+1}(X, \mathcal{O}_X) \cong H^{q+1}(\mathcal{A}_{0,\bullet}(X), \bar{\partial})$  are Hausdorff topological spaces.

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